Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Approximation Theory 129 (2004) 167–181
Journal of Approximation Theory
http://www.elsevier.com/locate/jat

The generalized decomposition theorem in Banach spaces and its applications ☆

Wen Song* and Zhengjun Cao[1]

Department of Mathematics, Harbin Normal University, Harbin 150080, PR China

Received 10 June 2003; accepted in revised form 14 June 2004

Communicated by Frank Deutsch

Abstract

In this paper, we first give a simple proof of the decomposition theorem in Alber (Field Inst. Comm. 25 (2000) 77) and then present a new decomposition of arbitrary elements in reflexive strictly convex and smooth Banach spaces. As applications of the decomposition theorem, we give the representations of the metric projection operator for some kind of closed convex sets. Finally, we provide a sufficient condition under which the generalized projection operator coincides with the metric projection operator.

Keywords: Duality mapping; Metric projection; Generalized projection; Basic variational principle; Moreau decomposition theorem; Generalized decomposition theorem

1. Introduction

It is well known that Moreau's decomposition theorem (see [12])

$$x = P_K x + P_{K^0} x, \quad \langle P_K x, P_{K^0} x \rangle = 0,$$

☆ This work was partially supported by the Excellent Young Teachers Program of MOE, P.R.C. and the Postdoctoral Grant of Heilongjiang province.
*Corresponding author.
E-mail address: wsong@hrbnu.edu.cn (W. Song).
[1] Current address: Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, PR China

0021-9045/$ - see front matter
doi:10.1016/j.jat.2004.06.002

where K is a closed convex cone in a Hilbert space and K^0 is the polar cone of K, is an important principle in Hilbert spaces. However, the transition from Hilbert space to Banach space is not so simple. In 1998, Alber [2] obtained a semi-definite decomposition of arbitrary elements of reflexive strictly convex and smooth Banach spaces by using the generalized projection, i.e.,

$$x = J^{-1}\Pi_{K^0}Jx + w, \quad \langle \Pi_{K^0}Jx, w \rangle = 0,$$

where K is a closed convex cone, $J : X \to X^*$ is the duality mapping, $w \in K$, and Π_{K^0} denotes the generalized projection operator. Later on, Alber [3] refined this result and obtained a definite decomposition by using the metric and the generalized projections, i.e.,

$$x = P_K x + J^{-1}\Pi_{K^0}Jx, \quad \langle \Pi_{K^0}Jx, P_K x \rangle = 0.$$

In this paper, we first give a simple proof of the decomposition theorem in [3] and then present a new decomposition of arbitrary elements in a reflexive strictly convex and smooth Banach space. As applications of the decomposition theorem, we give the representations of the metric projection operator for some kind of closed convex sets. Finally, we provide a sufficient condition under which the generalized projection operator coincides with the metric projection operator.

2. Preliminaries

Let X be a real Banach space with the dual space X^*. Denoting by $||\cdot||$ and $||\cdot||_*$ the norms on X and X^*, respectively. As usual, we denote the duality pairing of X^* and X by $\langle x^*, x \rangle$, or $\langle x, x^* \rangle$, where $x^* \in X^*$ and $x \in X$.

The duality mapping $J : X \to X^*$ defined by

$$J(x) = \{x^* \in X^* | \langle x^*, x \rangle = ||x^*||_*^2 = ||x||^2\} \quad \forall x \in X,$$

the duality mapping J^* from X^* to X is determined by

$$J^*(x^*) = \{x \in X \mid \langle x, x^* \rangle = ||x||^2 = ||x^*||_*^2\} \quad \forall x^* \in X^*.$$

The following basic results concerning the duality mapping are well known (see [6])

- X is reflexive if and only if J is surjective;
- X is smooth if and only if J is single-valued;
- X is strictly convex if and only if J is injective.

The definitions of the strict convexity, smoothness of Banach spaces and related properties can be found in [6,10].

If X is a reflexive Banach space, then X is strictly convex if and only if X^* is smooth and X is smooth if and only if X^* is strictly convex. In a word, if a Banach space X is reflexive strictly convex and smooth, then J, J^* are one-to-one single-valued operators and $J^{-1} = J^*$.

Let X be a Banach space, $C \subset X$ is a closed convex subset. The set-valued mapping $\pi(C\,|\cdot) : X \to C$ defined by

$$x \mapsto \pi(C\,|\cdot) = \{y \in C : ||x - y|| = d_C(x)\},$$

where $d_C(x) = \inf_{z \in C} ||x - z||$, is called the metric projection operator from X on C.

Note that for $x \in X$, $\pi(C|x)$ is the set of optimal solutions of the following minimization problem:

$$(\mathrm{P}_1) \quad \inf_{y \in C} ||x - y||^2.$$

The following characterization of the metric projection operator can be found in [9] (see also [16]).

Lemma 2.1. *Let C be a closed convex set of a Banach space X. Then*

$$\bar{x} \in \pi(C|x) \;\Leftrightarrow\; J(x - \bar{x}) \cap N(C, \bar{x}) \neq \emptyset,$$

where $N(C, \bar{x})$ is the normal cone to C at $\bar{x} \in C$ defined by

$$N(C, \bar{x}) := \{x^* \in X^* | \langle x - \bar{x}, x^* \rangle \leqslant 0 \quad \forall x \in C\}.$$

We recall that (see [15]): C is said to be proximinal if $\pi(C|x) \neq \emptyset$ for all $x \in X$; C is said to be a semi-Chebyshev set if $\pi(C|x)$ is a singleton at most; C is said to be a Chebyshev set if it is proximinal and semi-Chebyshev. It is known that [15] X is reflexive if and only if each closed convex subset of X is proximinal, and that X is strictly convex if and only if each closed convex subset of X is semi-Chebyshev. When $\pi(C\,|\cdot)$ is single-valued mapping, denoted by P_C, it is called the best approximate operator (metric projection operator).

In the following of this section, we assume that X is a reflexive strictly convex and smooth Banach space. In this situation, $\pi(C\,|\cdot) = P_C(\cdot)$ is single-valued, and Lemma 2.1 reduces to

$$\bar{x} = P_C(x) \;\Leftrightarrow\; \langle J(x - \bar{x}), y - \bar{x} \rangle \leqslant 0 \quad \text{for all } y \in C.$$

This inequality will be called the basic variational principle for P_C in X.

Consider now the following problem:

$$(\mathrm{P}_2) \quad \inf_{y \in C} W(x, y),$$

where $W(x, y) := ||x||^2 - 2\langle Jx, y \rangle + ||y||^2$ is called the *Lyapunov function*. We know that the problem (P_2) has an unique solution because $W(x, y)$ is strictly convex in y. The operator

$$\Pi_C x := \arg \min_{y \in C} W(x, y)$$

is said to be the *generalized projection* of x on C (see [1]).

Some applications of the generalized projection operator can be found in [1,2,5,8]. We recall (see [1]) that the following properties of the generalized projection operator on an arbitrary convex closed set C:

- The operator $\Pi_C : X \to C \subset X$ is identity on C, i.e., for every $x \in C$, $\Pi_C x = x$.
- The operator Π_C is a d-accretive operator in X, i.e.,

$$\langle Jx - Jy, \Pi_C x - \Pi_C y \rangle \geqslant 0 \quad \forall x, y \in X.$$

- The operator Π_C gives the absolutely best approximation of $x \in X$ relative to the functional $W(x, y)$, i.e.,

$$W(\Pi_C x, y) \leqslant W(x, y) - W(x, \Pi_C x) \quad \forall y \in C.$$

  Consequently, Π_C is the conditionally nonexpansive operator relative to the functional $W(x, y)$ in Banach spaces, i.e.,

$$W(\Pi_C x, y) \leqslant W(x, y) \quad \forall y \in C.$$

- In a Hilbert space H, J is an identity operator, $W(x, y) = ||x - y||^2$, Π_C coincides with the metric projection operator P_C.

The following result is of great importance (see [1]):

Lemma 2.2 (Basic variational principle for the generalized projection). *Assume that $C \subset X$ is a closed convex subset. Then $\hat{x} = \Pi_C x$ is the generalized projection of x on C if and only if the inequality*

$$\langle Jx - J\hat{x}, y - \hat{x} \rangle \leqslant 0 \quad \forall y \in C$$

holds.

Let K be a convex cone of X. Denoted by K^0 and K^+ the polar cone and the dual cone of K

$$K^0 = \{x^* \in X^* : \langle x^*, x \rangle \leqslant 0 \quad \forall x \in K\},$$

$$K^+ = \{x^* \in X^* : \langle x^*, x \rangle \geqslant 0 \quad \forall x \in K\}.$$

Obviously, K^0 and K^+ are closed convex cones in X^*; furthermore, $K^+ = -K^0$. If K is a closed convex cone, then $K^{00} = K$.

When C is a nonempty closed convex cone K, Lemmas 2.1 and 2.2 become

Corollary 2.1. *Let $K \subset X$ be a nonempty closed convex cone and $x \in X$. Then $\bar{x} = P_C(x)$ if and only if*

$$\langle J(x - \bar{x}), \bar{x} \rangle = 0$$

and

$$\langle J(x - \bar{x}), y \rangle \leqslant 0 \quad \forall y \in K.$$

Corollary 2.2. *Assume that $K \subset X$ is a nonempty closed convex cone. Then, for every $x \in X$, $\hat{x} = \Pi_K x$ if and only if*

$$\langle Jx - J\hat{x}, \hat{x} \rangle = 0$$

and

$$\langle Jx - J\hat{x}, y \rangle \leqslant 0 \quad \forall y \in K.$$

Corollary 2.3. *Assume that K is a nonempty closed convex cone of X. Then*

$$\Pi_K(x) = J^* \Pi_{K^+ + Jx}(J0)$$

and

$$||\Pi_K(x)|| \leqslant ||x|| \quad \forall x \in X.$$

By applying Corollary 2.1, one can obtain the decomposition of an arbitrary element x of X in the form (see [3,13]): there exists $v \in J^*(K^0)$ such that

$$x = P_K x + v \quad \text{and} \ \langle Jv, P_K x \rangle = 0.$$

Corollary 2.2 was used in [2] to obtain another type decomposition, namely, there exists $w \in K$ such that

$$x = J^* \Pi_{K^0} Jx + w \quad \text{and} \ \langle \Pi_{K^0} Jx, w \rangle = 0.$$

However, both decompositions are "semi-definite" because the elements v and w are unknown. By combining the metric projection and the generalized projection operators, Alber [3, Theorem 2.4] obtained a completely determined decomposition of arbitrary elements of X, namely

$$x = P_K x + J^* \Pi_{K^0} Jx \quad \text{and} \quad \langle \Pi_{K^0} Jx, P_K x \rangle = 0.$$

In the following section, we shall give a simple proof of the above decomposition theorem.

3. Generalized decomposition theorem

Theorem 3.1 (Alber [3, Theorem 2.4]). *Assume that X is a real reflexive strictly convex and smooth Banach space, the set $K \subset X$ is a nonempty closed convex cone. Then for every $x \in X$ and $\phi \in X^*$, the decompositions*

$$x = P_K x + J^* \Pi_{K^0} Jx \quad \textit{and} \quad \langle \Pi_{K^0} Jx, P_K x \rangle = 0,$$

$$\phi = P_{K^0} \phi + J \Pi_K J^* \phi \quad \textit{and} \quad \langle P_{K^0} \phi, \Pi_K J^* \phi \rangle = 0$$

hold.

Proof. By Theorem 2.6 in [2], there exist $\omega \in K$ and $\psi \in K^0$ such that

$$x = J^* \Pi_{K^0} Jx + \omega \quad \text{and} \quad \langle \Pi_{K^0} Jx, \omega \rangle = 0, \tag{1}$$

$$\phi = J \Pi_K J^* \phi + \psi \quad \text{and} \quad \langle \psi, \Pi_K J^* \phi \rangle = 0. \tag{2}$$

We next prove that ω and ψ in (1) and (2) are just $P_K x$ and $P_{K^0}\phi$, respectively. From (1), we have $\langle \Pi_{K^0} Jx, x \rangle = \langle \Pi_{K^0} Jx, x - \omega \rangle = ||x - \omega||^2$. Hence, for every $y \in K$,

$$\langle \Pi_{K^0} Jx, x - y \rangle = ||x - \omega||^2 - \langle \Pi_{K^0} Jx, y \rangle \geqslant ||x - \omega||^2.$$

The last inequality comes from $y \in K$, $\Pi_{K^0} Jx \in K^0$. Thus,

$$\begin{aligned} ||x - \omega||^2 &\leqslant ||\Pi_{K^0} Jx||_* ||x - y|| \\ &= ||J(x - \omega)||_* ||x - y|| = ||x - \omega|| \, ||x - y||. \end{aligned}$$

If $||x - \omega|| = 0$, then $x \in K$ and $\omega = P_K x$. If $||x - \omega|| > 0$, then

$$||x - \omega|| \leqslant ||x - y|| \quad \forall y \in K,$$

i.e., $\omega = P_K x$. The second part can be proved similarly. $\square$

Remark. When X is a Hilbert space, the decomposition $x = P_K x + J^* \Pi_{K^0} Jx$, reduces to

$$x = P_K x + P_{K^0} x,$$

which is just the Moreau decomposition theorem [12]. If K is a subspace L in a Hilbert space H and $L^\perp$ is its orthogonal complement, then this theorem reduces to the Risz decomposition, i.e.,

$$x = P_L x + P_{L^\perp} x.$$

In 2001, Wang and Wang [13] obtained a generalized orthogonal decomposition theorem for a proximinal (Chebyshev) linear subspace in Banach spaces. In the following, we shall extend it to closed convex cones in Banach spaces. For each $x \in X$, set $x_1^\perp = \{x^* \in X^* | \langle x, x^* \rangle = 0\}$.

Theorem 3.2. *Let X be a Banach space, $K \subset X$ be a proximinal closed convex cone. Then, for every $x \in X$,*

$$x = x_1 + x_2, \quad x_1 \in K, \quad x_2 \in J^{-1}(K^0) = \pi(K|\cdot)^{-1}(0) \quad \textit{and} \quad J(x_2) \cap x_1^\perp \neq \emptyset.$$

If K is Chebyshev, then

$$x = P_K x + x_2, \quad x_2 \in J^{-1}(K^0 \cap (P_K x)^\perp)$$

and the decomposition is unique.

Proof. By Lemma 2.1 we easily see that $J^{-1}(K^0) = \pi(K|\cdot)^{-1}(0)$.

Since K is proximinal, $\pi(K|x) \neq \emptyset$ for all $x \in X$. Take $x_1 \in \pi(K|x)$, and let $x_2 = x - x_1$. Then $x = x_1 + x_2$ and

$$J(x_2) \cap N_K(x_1) = J(x - x_1) \cap N_K(x_1) \neq \emptyset$$

by Lemma 2.1. Since K is a closed convex cone, we see that $N_K(x_1) = K^0 \cap x_1^\perp$. It follows that $x_2 \in J^{-1}(K^0)$ and $J(x_2) \cap x_1^\perp \neq \emptyset$.

If K is Chebyshev, then, for every $x \in X \backslash K$, $\pi(K|x)$ is singleton, denoted by $P_K(x)$. Hence

$$x = P_K x + x_2, \quad x_2 \in J^{-1}(K^0 \cap (P_K x)^{\perp}).$$

Suppose that there is another decomposition for x, for instance,

$$x = u_1 + u_2, \quad u_1 \in K, \quad u_2 \in J^{-1}(K^0 \cap u_1^{\perp}).$$

Then

$$J(x - u_1) \cap N_K(u_1) = J(u_2) \cap N_K(u_1) \neq \emptyset,$$

and then $u_1 \in \pi(K|x)$ by Lemma 2.1. Since K is Chebyshev, we have $u_1 = P_K(x)$. Hence $u_2 = x - P_K(x) = x_2$. This shows that the decomposition is unique. $\square$

When K is a linear subspace, $K^0 = K^{\perp}$ and $K^{\perp} \cap x_1^{\perp} = K^{\perp}$. Hence, in this case, the theorem above reduces to Theorem 3.2 in [13].

The question arises: whether we can obtain a decomposition result as in Theorem 3.1 under the assumptions of Theorem 3.2?

Corollary 3.1. *Let X be a reflexive strictly convex Banach space, $K \subset X$ be a closed convex cone. Then, for every $x \in X$, the (unique) decomposition*

$$x = P_K x + x_2, \ x_2 \in J^{-1}(K^0) \quad \text{and} \quad \langle J(x_2), P_K(x) \rangle = 0$$

holds.

In fact, we can derive Theorem 3.1 from Corollary 3.1. In the following, we will discuss another decomposition of arbitrary elements in a reflexive strictly convex and smooth Banach space.

Lemma 3.1. *Let X be a real reflexive strictly convex and smooth Banach space and let $f(\cdot)$ be a finite convex function on X. Suppose that $z \in X$ and $C := \{x \in X : f(x) \leqslant t\}$ is the sub-level set of $f(\cdot)$ corresponding to t with $f(z) > t > \inf f(X)$. Then*

$$\Pi_C z = (J + \mu\partial f)^{-1}(Jz),$$

where μ is an arbitrary positive solution to

$$f((J + \mu\partial f)^{-1}(Jz)) = t.$$

Proof. We know that $\Pi_C z$ is the unique solution of the convex program

$$\min\{\tfrac{1}{2}||z||^2 - \langle Jz, y \rangle + \tfrac{1}{2}||y||^2\} \quad \text{subject to } f(y) \leqslant t.$$

The Slater's condition holds for this program. Hence, there exists a Lagrange multiplier $\mu \geqslant 0$ with

$$Jz \in J\Pi_C z + \mu\partial f(\Pi_C z) \quad \text{and} \quad \mu(f(\Pi_C z) - t) = 0.$$

Since $z \notin C$, we conclude $\mu > 0$. Thus, $f(\Pi_C z) = t$ and

$$Jz \in (J + \mu \partial f)(\Pi_C z).$$

Since the resolvent $(J + \mu\partial f)^{-1}$ of a maximal monotone operator ∂f is singled-valued, we have

$$\Pi_C z = (J + \mu \partial f)^{-1}(Jz). \qquad \square$$

Theorem 3.3. *Let C be a nonempty bounded closed convex subset of a real reflexive strictly convex and smooth Banach space X and let*

$$C^0 := \{x^* \in X^* : \langle x^*, x \rangle \leqslant 1, \ x \in C\}$$

be the polar set of C. Then for every $y^* \in X^* \backslash C^0$ and $z \in X$ with $Jz \notin C^0$,

$$J^* y^* = \mu P_C\left(\frac{J^* y^*}{\mu}\right) + J^*(\Pi_{C^0} y^*),$$

where $\mu = \langle \Pi_{C^0} y^*, \ J^* y^* - J^*(\Pi_{C^0} y^*) \rangle$ is the unique positive solution of

$$\mu = \left\langle J\left(J^* y^* - \mu P_C\left(\frac{J^* y^*}{\mu}\right)\right), \mu P_C\left(\frac{J^* y^*}{\mu}\right)\right\rangle$$

and

$$z = \mu P_C\left(\frac{z}{\mu}\right) + J^*(\Pi_{C^0} Jz),$$

where $\mu = \langle \Pi_{C^0} Jz, \ z - J^* \Pi_{C^0} Jz \rangle$ is the unique positive solution of

$$\mu = \left\langle J\left(z - \mu P_C\left(\frac{z}{\mu}\right)\right), \mu P_C\left(\frac{z}{\mu}\right)\right\rangle.$$

Proof. The polar set C^0 is nothing but the sublevel set of the continuous convex finite (since C is bounded) function δ_C^* corresponding to 1 (where δ_C denotes the indicator function of C and δ_C^* is the conjugate of δ_C). Since $\delta_C^*(y^*) > 1 > 0 = \delta_C^*(0)$. We apply Lemma 3.1 for y^*

$$\Pi_{C^0} y^* = (J^* + \mu \partial \delta_C^*)^{-1}(J^* y^*),$$

where μ is an arbitrary positive solution to the equation

$$\delta_C^*((J^* + \mu \partial \delta_C^*)^{-1}(J^* y^*)) = \delta_C^*(\Pi_{C^0} y^*) = 1.$$

Further, $\Pi_{C^0} y^* = (J^* + \mu \partial \delta_C^*)^{-1}(J^* y^*)$ if and only if $J^* y^* \in (J^* + \mu \partial \delta_C^*)(\Pi_{C^0} y^*)$, i.e., $\frac{J^* y^* - J^*(\Pi_{C^0} y^*)}{\mu} \in \partial \delta_C^*(\Pi_{c^0} y^*)$, equivalently, $\Pi_{C^0} y^* \in \partial \delta_C\left(\frac{J^* y^* - J^*(\Pi_{C^0} y^*)}{\mu}\right)$. Since $\partial \delta_C\left(\frac{J^* y^* - J^*(\Pi_{C^0} y^*)}{\mu}\right)$ is a convex cone, the above inclusion is equivalent to

$$\frac{\Pi_{C^0} y^*}{\mu} \in \partial \delta_C\left(\frac{J^* y^* - J^*(\Pi_{C^0} y^*)}{\mu}\right).$$

That is

$$JJ^*\left(\frac{\Pi_{C^0}y^*}{\mu}\right)\in\partial\delta_C\left(\frac{J^*y^*-J^*(\Pi_{C^0}y^*)}{\mu}\right),$$

which is equivalent to

$$\left\langle J\left(\frac{J^*y^*}{\mu}-\frac{J^*y^*-J^*(\Pi_{C^0}y^*)}{\mu}\right), y-\frac{J^*y^*-J^*(\Pi_{C^0}y^*)}{\mu}\right\rangle\leqslant 0 \quad \text{for all } y\in C.$$

It follows from Lemma 2.1 that

$$\frac{J^*y^*-J^*(\Pi_{C^0}y^*)}{\mu}=P_C\left(\frac{J^*y^*}{\mu}\right).$$

Therefore,

$$J^*y^*=\mu P_C\left(\frac{J^*y^*}{\mu}\right)+J^*(\Pi_{C^0}y^*).$$

Since $\delta_C^*(\Pi_{C^0}y^*)=1$, we have

$$\begin{aligned}1&=\delta_C^*(\Pi_{C^0}y^*)+\delta_C\left(P_C\left(\frac{J^*y^*}{\mu}\right)\right)\\&=\delta_C^*(\Pi_{C^0}y^*)+\delta_C\left(\frac{J^*y^*-J^*(\Pi_{C^0}y^*)}{\mu}\right)\\&=\left\langle\Pi_{C^0}y^*,\frac{J^*y^*-J^*(\Pi_{C^0}y^*)}{\mu}\right\rangle.\end{aligned}$$

Hence

$$\mu=\langle\Pi_{C^0}y^*,\ J^*y^*-J^*(\Pi_{C^0}y^*)\rangle.$$

On the other hand, if we replace y^* with Jz, then we complete the proof. $\square$

4. The representative of the metric projection

From Theorem 3.1, one sees that $P_Kx=x-J^*\Pi_{K^0}Jx$. That is to say that the metric projection P_Kx can be obtained by calculating $J^*\Pi_{K^0}Jx$. We will find, in many cases, that the calculation of the generalized projection can be reduced to calculate the minimum of a quadratic function with one variable. Our results generalize the corresponding results in [7] from Hilbert spaces to reflexive, strictly convex and smooth Banach spaces.

Theorem 4.1. *Suppose that X is a real reflexive strictly convex and smooth Banach space, $x_0^*\in X^*\backslash\{0\}$ and $c\in R$. Let $L=\{x\in X:\langle x_0^*,x\rangle=0\}$ and*

$H_c = \{x \in X : \langle x_0^*, x \rangle = c\}$. *Then*

$$P_L(x) = x - \frac{\langle x_0^*, x \rangle}{\|x_0^*\|_*^2} J^*(x_0^*) \quad \forall x \in X,$$

$$P_{H_c}(x) = x - \frac{\langle x_0^*, x \rangle - c}{\|x_0^*\|_*^2} J^*(x_0^*) \quad \forall x \in X.$$

Proof. According to Theorem 3.1, we need only to calculate $\Pi_{L^\perp} Jx$. Note that $L^\perp = \{rx_0^* : r \in R\}$. By Proposition 1.3 in [3], we have $\Pi_{L^\perp} Jx = \frac{\langle x_0^*, x \rangle}{\|x_0^*\|_*^2} x_0^*$. Hence

$$P_L x = x - J^*\left(\frac{\langle x_0^*, x \rangle}{\|x_0^*\|_*^2} x_0^*\right) = x - \frac{\langle x_0^*, x \rangle}{\|x_0^*\|_*^2} J^*(x_0^*).$$

We know that there exists an element $y_0 \in X$ such that $H_c = L + \{y_0\}$, where $x_0^*(y_0) = c$. Hence, we have

$$\begin{aligned} P_{H_c}(x) &= P_{L+y_0}(x) = P_L(x - y_0) + y_0 \\ &= \left[(x - y_0) - \frac{\langle x_0^*, x - y_0 \rangle}{\|x_0^*\|_*^2} J^*(x_0^*)\right] + y_0 \\ &= x - \frac{\langle x_0^*, x \rangle - c}{\|x_0^*\|_*^2} J^*(x_0^*). \qquad \square \end{aligned}$$

Remark. Theorem 4.1 was proved in [14] by using different methods.

If $K = \{x \in X \colon \langle x_0^*, x \rangle \leqslant 0\}$, where $x_0^* \in X^* \backslash \{0\}$, then $K^0 = \{rx_0^* : r \geqslant 0\}$ and the representatives of the metric projection can be obtained as well.

Theorem 4.2. *Suppose that X is a real reflexive strictly convex and smooth Banach space, $x_0^* \in X^* \backslash \{0\}$ and $c \in R$. Let $K = \{x \in X : \langle x_0^*, x \rangle \leqslant 0\}$ and $K_c = \{x \in X : \langle x_0^*, x \rangle \leqslant c\}$. Then for each $x \in X$,*

$$P_K x = x - \max\left\{0, \frac{\langle x_0^*, x \rangle}{\|x_0^*\|_*^2}\right\} J^*(x_0^*)$$

and

$$P_{K_c} x = x - \max\left\{0, \frac{\langle x_0^*, x \rangle - c}{\|x_0^*\|_*^2}\right\} J^*(x_0^*).$$

Proof. Since $K^0 = \{rx_0^* : r \geqslant 0\}$, we have

$$W(Jx, \Pi_{k^0} Jx) = \min_{r \geqslant 0}\{\|x\|^2 - 2r\langle x, x_0^* \rangle + r^2 \|x_0^*\|_*^2\}.$$

Obviously, the minimum is attained at $r = \max\{0, \frac{\langle x_0^*, x\rangle}{\|x_0^*\|_*^2}\}$. Hence,

$$\Pi_{K^0} Jx = \max\left\{0, \frac{\langle x_0^*, x\rangle}{\|x_0^*\|_*^2}\right\} x_0^*.$$

By Theorem 3.1, we have

$$P_K x = x - J^* \Pi_{K^0} Jx = x - \max\left\{0, \frac{\langle x_0^*, x\rangle}{\|x_0^*\|_*^2}\right\} J^*(x_0^*). \qquad \square$$

We recall that

$$C_\alpha := \{(x, r) \in X \times R : \|x\| \leqslant \alpha r\},$$

where $\alpha > 0$, is said to be an *icecream cone* of $X \times R$. The polar cone of C_α is

$$C_\alpha^0 := \{(x^*, s) \in X^* \times R : \langle (x^*, s), (x, r)\rangle = x^*(x) + sr \leqslant 0 \quad \forall (x, r) \in C_\alpha\}.$$

Since the nonpolyhedral structure of the icecream cone, the explicit formula for the projection on it is very important. Some other results related to the icecream cone can be found in [11].

Theorem 4.3. *Let X be a real reflexive strictly convex and smooth Banach space and let C_α be an ice cream cone in $X \times R$. Then $-C_\alpha^0 = (J \times I)(C_{1/\alpha}) = \{(x^*, s) \in X^* \times R \mid \|x^*\|_* \leqslant s/\alpha\}$, and for every $(x, r) \in X \times R$,*

$$P_{C_\alpha}(x, r) = \begin{cases} (x, r) & \text{if } \|x\| \leqslant \alpha r, \\ (0, 0) & \text{if } \|\alpha x\| \leqslant -r, \\ \dfrac{\alpha\|x\| + r}{\alpha^2 + 1}\left(\alpha \dfrac{x}{\|x\|}, 1\right) & \text{otherwise.} \end{cases}$$

Proof. By the definition of duality mapping J, it is easy to see that $(J \times I)(C_{1/\alpha}) = \{(x^*, s) \in X^* \times R \mid \|x^*\|_* \leqslant s/\alpha\}$. We will show that $-C_\alpha^0 \subset (J \times I)(C_{1/\alpha})$. Take any $(x^*, s) \in C_\alpha^0$. Then $(\alpha J^* x^*, \|x^*\|_*) \in C_\alpha$. Hence

$$\langle (x^*, s), (\alpha J^* x^*, \|x^*\|_*)\rangle = \alpha \|x^*\|_*^2 + s\|x^*\|_* \leqslant 0.$$

If $\|x^*\|_* \neq 0$, then $\|x^*\|_* \leqslant -s/\alpha$, equivalently $(-x^*, -s) \in (J \times I)(C_{1/\alpha})$. Suppose $x^* = 0$. Since $(0, 1) \in C_\alpha$ and $(0, s) \in C_\alpha^0$, we have $s \leqslant 0$. Hence, $\|x^*\| = 0 \leqslant -s/\alpha$ and hence $(0, -s) \in (J \times I)(C_{1/\alpha})$. We next show that $-(J \times I)(C_{1/\alpha}) \subset C_\alpha^0$. Take any $(x^*, s) \in -(J \times I)(C_{1/\alpha})$, i.e., $\|x^*\|_* \leqslant -s/\alpha$. For every $(x, r) \in C_\alpha$, we have

$$\langle (x^*, s), (x, r)\rangle = \langle x^*, x\rangle + rs \leqslant \alpha r \times (-s/\alpha) + rs = 0.$$

By the definition of polar cone, we have $-(J \times I)(C_{1/\alpha}) \subset C_\alpha^0$.

By Theorem 3.1, it suffices to calculate $\Pi_{C_\alpha^0}(Jx, r)$. The first case is obvious. If $\|\alpha x\| \leqslant -r$, then $(Jx, r) \in C_\alpha^0$. Hence $\Pi_{C_\alpha^0}(Jx, r) = (Jx, r)$ and hence

$(J^* \times I)(\Pi_{C_\alpha^0}(Jx, r)) = (x, r)$. If $||x|| > \alpha r$ and $||\alpha x|| > -r$, we need show that

$$\Pi_{C_\alpha^0}(Jx, r) = \frac{||x|| - \alpha r}{\alpha^2 + 1}\left(\frac{Jx}{||x||}, -\alpha\right).$$

Let us abbreviate $r_0 := \frac{\alpha^2 r - \alpha||x||}{1+\alpha^2}$, $x_0^* = -\frac{r_0}{\alpha}\frac{Jx}{||x||}$. Then the formula above becomes

$$\Pi_{C_\alpha^0}(Jx, r) = (x_0^*, r_0).$$

It is clear that $r_0 < 0$ and $(x_0^*, r_0) \in C_\alpha^0$. We check the basic variational inequality for generalized projection: Take any $(x^*, s) \in C_\alpha^0$

$$\begin{aligned}
&\langle (J^* \times I)(Jx, r) - (J^* \times I)(x_0^*, r_0), (x^*, s) - (x_0^*, r_0) \rangle \\
&\quad = \langle x^* - x_0^*, x - J^* x_0^* \rangle + (r - r_0)(s - r_0) \\
&\quad \leqslant (||x^*||_* + ||x_0^*||_*)||x - J^* x_0^*|| + (r - r_0)(s - r_0) \\
&\quad \leqslant \left(\frac{r_0}{\alpha} - \frac{s}{\alpha}\right)(||x|| + ||J^* x_0^*||) + (r - r_0)(s - r_0) \\
&\quad = \left(\frac{r_0}{\alpha} - \frac{s}{\alpha}\right)\left(||x|| + \frac{r_0}{\alpha}\right) + (r - r_0)(s - r_0) \\
&\quad = (r_0 - s)\left(\frac{\alpha||x|| + r_0}{\alpha^2} - r + r_0\right) = 0. \qquad \square
\end{aligned}$$

5. A sufficient condition for the coincidence of the generalized projection and the metric projection

It should be observed that, in general, the metric projection and the generalized projection do not coincide. The following example provided in [4] illustrates this fact.

Example 1. Let $X = \mathbb{R}^3$ be endowed with the norm

$$||(x_1, x_2, x_3)|| = (x_1^2 + x_2^2)^{1/2} + (x_2^2 + x_3^2)^{1/2}.$$

This is a strictly convex and smooth Banach space and $K = \{x \in \mathbb{R}^3 \mid x_2 = x_3 = 0\}$ is a closed convex cone of it. Simple computation show that $P_K(1,1,1) = (1,0,0)$ and $\pi_K(1,1,1) = (2,0,0)$.

In the following, we shall provide a condition which ensures that the generalized projection $\Pi_K x$ coincides with the metric projection $P_K x$.

Theorem 5.1. *Suppose that X is a real reflexive strictly convex and smooth Banach space, $K \subset X$ is a nonempty closed convex cone, and that $x \in X$ satisfies*

$$\langle P_K x, Jx - JP_K x \rangle = 0. \tag{3}$$

Then

$$P_K x = \Pi_K x.$$

Proof. From (3), we have

$$\langle J^*(Jx - (Jx - JP_K x)), Jx - JP_K x \rangle = \langle P_K(x), Jx - JP_K x \rangle = 0.$$

It is clear that

$$\langle J^*(Jx - (Jx - JP_K x)), x^* \rangle = \langle P_K x, x^* \rangle \leqslant 0 \quad \text{for all } x^* \in K^0.$$

It follows from Corollary 2.1 that

$$P_{K^0} Jx = Jx - JP_K x.$$

Hence

$$Jx = JP_K x + P_{K^0} Jx.$$

On the other hand, by Theorem 3.1, we also have

$$Jx = J\Pi_K J^* Jx + P_{K^0} Jx = J\Pi_K x + P_{K^0} Jx.$$

Since the decomposition is unique and J is injective, we have

$$P_K x = \Pi_K x. \qquad \square$$

Remark. If $x \in X$ satisfies

$$\langle Jx - J(x - J^* \Pi_{K^0} Jx), x - J^* \Pi_{K^0} Jx \rangle = 0$$

and

$$Jx - J(x - J^* \Pi_{K^0} Jx) \in K^0,$$

then $P_K x = \Pi_K x$. This assertion follows from Theorems 3.1 and 2.9 in [2].

Corollary 5.1. *Suppose that X is a real reflexive strictly convex and smooth Banach space, $x, y \in X$, $y \neq 0$ and that $K = \{ry : r \in R\}$. If $P_{K^0} Jx = \Pi_{K^0} Jx$, then*

$$\langle Jx, y \rangle \langle Jy, x \rangle \geqslant 0.$$

Proof. By Proposition 1.3 in [3],

$$\Pi_K x = \frac{\langle Jx, y \rangle}{\|y\|^2} y.$$

By Theorem 3.1, one has

$$Jx = J\Pi_K J^* Jx + P_{K^0} Jx = J\left(\frac{\langle Jx, y \rangle}{\|y\|^2} y \right) + P_{K^0} Jx.$$

Hence,

$$\langle Jx, x \rangle = \langle P_{K^0} Jx, x \rangle + \left\langle \frac{\langle Jx, y \rangle}{\|y\|^2} Jy, x \right\rangle$$

and hence

$$\|x\|^2 - \frac{\langle Jx, y\rangle}{\|y\|^2}\langle Jy, x\rangle = \langle \Pi_{K^0} Jx, x\rangle$$
$$\leqslant \|Jx\|_*\|x\| = \|x\|^2.$$

The last step comes from the Corollary 2.3. Thus,

$$\frac{\langle Jx, y\rangle}{\|y\|^2}\langle Jy, x\rangle \geqslant 0,$$

i.e.,

$$\langle Jx, y\rangle \langle Jy, x\rangle \geqslant 0. \qquad \square$$

Acknowledgments

The authors are grateful to the referee for helpful comments on improving the first version of this paper

References

[1] Ya.I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: A. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Monotone and Accretive Type, Marcel Dekker, New York, 1996, pp. 15–50.
[2] Ya.I. Alber, Generalized projections decompositions and the Pythagoream-type theorem in Banach spaces, Appl. Math. Lett. 11 (1998) 115–121.
[3] Ya.I. Alber, Decomposition theorem in Banach spaces, Field Inst. Comm. 25 (2000) 77–93.
[4] Y. Alber, D. Butnariu, Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces, J. Optim. Theory Appl. 92 (1997) 33–61.
[5] Ya.I. Alber, S. Gurre-Delabriere, On the projection methods for fixed point theorems, Analysis 21 (2001) 17–39.
[6] V. Barbu, Th. Precupanu, Convexity and Optimization in Banach Spaces, Romania International Publishers, Bucuresti, 1978.
[7] H.H. Bauschke, Projection algorithms and monotone operators, Ph.D. Thesis, Simon Fraser University, 1996.
[8] C.E. Chidume, M. Khumalo, H. Zegeye, Generalized projection and approximation of fixed points of nonself maps, J. Approx. Theory 120 (2003) 242–252.
[9] F.R. Deutsch, P.H. Maserick, Applications of the Hahn–Banach theorem in approximation theory, SIAM Rev. 9 (1967) 516–530.
[10] J. Diestel, Geometry of Banach Spaces, Lecture Notes in Mathematics, Springer, New York, 1975.
[11] R. Loewy, H. Scheider, Positive operators on the n-dimensional ice cream cone, J. Math. Anal. Appl. 49 (1975) 375–392.
[12] J.J. Moreau, Decomposition orthogonal dun espace Hilbertien selon deux cones mutuellement polaires, C. R. Acad. Sci. Paris 255 (1962) 238–240.

[13] Yuwen Wang, Hui Wang, Generalized orthogonal decomposition theorem in Banach spaces and generalized orthogonal complemented subspaces, Acta Math. Sinica 44 (6) (2001) 1045–1050 (in Chinese).

[14] Yuwen Wang, Jinfeng Yu, The characterization and representative of a class of metric projection in Banach Space, Acta Math. Sci. 21A (2001) 29–35 (in Chinese).

[15] Shiying Xu, Chong Li, Wenshan Yang. Non-Linear Approximation Theory in Banach Spaces, Scientific Press, Beijing, 1997 (in Chinese).

[16] C. Zalinescu, Convex Analysis in General Vector Spaces, World Scientific, Singapore, 2002.